SENSITIVITY-BASED MODEL UPDATING USING MULTIPLE TYPES OF SIMULTANEOUS STATE VARIABLES

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ABSTRACT

Model tuning methods based on sensitivity coefficients offer the advantage that many different types of state variables (reference responses) and design variables can be used. However, combining different types of state and design variables in a single sensitivity matrix may lead to stability problems in the updating algorithm. This problem is demonstrated when combining resonance frequencies and modal displacements as state variables in an objective function. The stability of the solution can be guaranteed by appropriate scaling of the sensitivity matrix also leading to faster convergence. Various scaling methods are explored from a statistical parameter estimation point of view. The results are illustrated with application examples.

INTRODUCTION

There are two main approaches to finite element model updating: (i) model matrix updating based on deterministic optimization methods like proposed by Baruch [1] and Berman, Nagy [2] and (ii) sensitivity-based model updating using statistical parameter identification. The deterministic approach will not be further considered in this paper.

The general form of the objective function used in sensitivity-based model updating is a weighted sum of Euclidean norms:

$$J(p) = k_y \left\| \Delta y^T W_y \Delta y \right\| + k_p \left\| \Delta p^T W_p \Delta p \right\| \tag{1.a}$$

$$\{\Delta p\} = \{p_a - p_o\} \quad \{\Delta y\} = \begin{Bmatrix} \lambda_a - \lambda_e \\ \varphi_a - \varphi_e \end{Bmatrix} \tag{1.b}$$

where the n-dimensional vector $\{p_a\}=\{p_{1a},...,p_{na}\}$ are the parameter values to be updated, $\{p_o\}=\{p_{1o},...,p_{no}\}$ are initial parameter values, $\{\lambda_a\}=[\lambda_{1a},...,\lambda_{pa}]^T$, $\{\varphi_a\}=[\varphi_{1a},...\varphi_{pa}]^T$ are analytical eigenvalues and eigenvectors, $\{\lambda_e\}=[\lambda_{1e},...,\lambda_{qe}]^T$, $\{\varphi_e\}=[\varphi_{1e},...,\varphi_{qe}]^T$ are experimental eigenvalues and eigenvectors (with dimension p+q=m), *Wy* and *Wp* are positive definite and usually symmetric weighting matrices. *ky* and *kp* are the weighting coefficients for the two terms. The solution of (1) is called the Bayesian estimator.

A special case of (1) is to set *kp* to zero and *ky*=1

$$J(p) = \Delta y^T W_y \Delta y \tag{2}$$

The solution of (2) is the weighted least square solution.

Sensitivity based model updating can be further classified into two sub-categories depending on how parameters are defined: (i) Submatrix Updating and (ii) Physical Parameter Updating.

Submatrix Updating

The matrix updating method assumes that

$$[K] = [K_a] + \sum \alpha_i [K_i] \tag{3.a}$$

$$[M] = [M_a] + \sum \beta_i [M_i] \tag{3.b}$$

where αi and βi are the unknown parameters to be updated. *Ki* and *Mi* are correction submatrices, related to elements or element groups defining possible error sources [3, 4].

$$\{p\} = \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \tag{3.c}$$

and $\{\alpha\} = [\alpha_1,\alpha_2,...]^T$, $\{\beta\}=[\beta_1,\beta_2,...]^T$

Physical Parameter Updating

From all the design variables in a finite element model, any combination of physical properties can be selected as parameters for the model updating algorithm. A parameter defined in this way is clearly related to an element or a group of elements. Because of the physical meaning, an updated value can be easily interpreted and evaluated against criteria set by the analyst. This explains why this approach has become more popular and has been chosen for implementation in commercial software programs [9].

In earlier applications, only eigenvalues were selected as response (state variables) that needed to satisfy the correlation requirement, although modal displacements have since long time been proposed as responses [5]. The problem with using mode shape data as responses is not only the incompleteness of the experimental data, but mainly that the sensitivity matrix tends to become ill-conditioned.

Lallement et al. [6] proposed a method in which the most sensitive parameters are selected one by one according to their influence on the rank and condition number of the sensitivity matrix. However, this is a very expensive method to use. Alternative solutions based on scaling of the sensitivity matrix by acting on the weighting matrices *Wp* will be discussed hereafter.

In order to achieve more reliable analytical models, the rigid body characteristics like total mass, moments of inertia, and center of gravity, should also be taken into account. This idea was also proposed by Caesar [7].

Using MAC as response to improve model updating is proposed by Heylen et al [8]. The sensitivity of MAC with respect to a certain parameter can easily be obtained if the mode shape derivatives are known. [9].

Other possible responses which can be used to extend the measurement database, like for example force residuals, orthogonality conditions, etc. have been proposed by different authors. From a statistical parameter estimation point of view, the more measurement data, the more accurate the parameter estimation will be. On the other side, when simultaneously using different response types, numerical ill-conditioning due to large sensitivity discrepancies is more likely to occur.

Some general methods that can be used to improve the numerical condition of sensitivity matrices by acting on *Wy* are further investigated in this paper.

BAYESIAN PARAMETER ESTIMATION

The stochastic form of the Bayesian parameter estimation model is given by [10, 11, 13]

$$\{y\}-\{y_o\}=\frac{\partial y}{\partial p}(\{p\}-\{p_o\}) \quad (4)$$

$$\{y_e\}=\{y\}+\{\varepsilon\} \quad (5)$$

Setting

$$\{\Delta y\}=\{y_e\}-\{y\},\{\Delta p\}=\{p\}-\{p_o\} \quad (6)$$

the statistical properties (mean and variance) of {Δy} and {Δp} are assumed to be normal [5]

$$E(\Delta y)=E(\Delta p)=0,E(\{\varepsilon\}\{\Delta p\}^T)=0 \quad (7)$$

$$E(\{\varepsilon\}\{\varepsilon\}^T)=V_y,E(\{\Delta p\}\{\Delta p\}^T)=V_p \quad (8)$$

In the mathematical model, {ε} is the measurement error which is assumed to be white Gaussian (diagonal covariance matrix) and uncorrelated with {Δ*p*}. Iteration starts from the initial parameter and covariance matrices and stops when exact {y} is obtained ({Δ*y*} -> {ε}), or within a certain accuracy. The joint probability density function of observation {Δ*y*} and {Δ*p*} can be written as [13]

$$P(\Delta p,\Delta y)=\frac{1}{(2\pi)^{\frac{n+m}{2}}\sqrt{V_pV_y}}e^{-\frac{1}{2}(\Delta p^TW_p\Delta p+\Delta y^TW_y\Delta y)} \quad (9.a)$$

$$W_p=V_p^{-1},W_y=V_y^{-1} \quad (9.b)$$

Wp is given by previous iteration, *Wy* is independent of p. Maximizing the joint probability density function is equivalent to minimizing an error function (1.a). Minimization of (1.a) or maximization of (9.a) leads to the following updating algorithm

$$\{p\}=\{p_o\}+[G]\{\Delta y\} \quad (10a)$$

$$[G]=[W_p]^{-1}[S]^T([S][W_p]^{-1}[S]^T+[W_y]^{-1})^{-1} \quad (10b)$$

$$[W_p^*]^{-1}=([I]-[G][S])[W_p]^{-1} \quad (10c)$$

where [*S*]=∂y/∂p is the sensitivity matrix and *Wp** is the estimation error covariance. For a linear model and assuming a normal distribution, the estimation is a minimum variance estimation [10,11,12].

Using the modified Sherman-Morrison-Woodbury formula [13], equation (10) can be written in an alternate form as

$$\{p\}=\{p_o\}+[G]\{\Delta y\} \quad (11a)$$

$$[G]=([S]^T[W_y][S]+[W_p])^{-1}[S]^T[W_y] \quad (11b)$$

$$[W_p^*]=([S]^T[W_y][S]+[W_p])^{-1} \quad (11c)$$

From (10) and (11), it can be found that if measurement error is small, *Wy* will be larger, then gain matrix [G] will be large too, and more modification will be made to the previous estimation

If confidence in the previous estimation of parameters is low, [G] will be large, and more modification will be made to the parameters. It works as an adaptive estimator.

Based on subjective guesses of the initial statistical properties, i.e. mean values and covariance of the parameters, using adaptive characteristics to modify a previous estimation, is the philosophy of the Bayesian estimator.

Convergence Criterion

As covariance matrices are positive definite,the matrices in (10.b) and (11.b) can always be inverted. Physically, the measurement data is more or less noise corrupted. But, this does not imply convergence of the algorithm. If convergence can be obtained, the necessary and sufficient condition for a linear system is obvious: the estimation error should reduce monotonically.

This means that

$$\|W_p^*(k)/W_p^*(k-1)\|<1 \quad (12a)$$

The equivalent expression is

$$\|([I]-[G][S])\|<1 \quad (12b)$$

(12) is applicable in a small linear range for a non-linear system. A more detailed mathematical description can be found in [12,14].

NORMALIZATION OF PARAMETERS AND RESPONSES

The simplest form of scaling which is often very effective is to normalize the parameters and measurement data by their initial values. Define the scaling matrices [D_p] and [D_y] as

$$[D_p]=\begin{bmatrix} 1/p_1 & & 0 \\ & . & \\ 0 & & 1/p_n \end{bmatrix},[D_y]=\begin{bmatrix} 1/y_1 & & 0 \\ & . & \\ 0 & & 1/y_n \end{bmatrix} \tag{13}$$

The scaled parameter variation and scaled measurement data are

$$\{\Delta\bar{p}\}=[D_p]\{\Delta p\},\{\Delta\bar{y}\}=[D_y]\{\Delta y\} \tag{14}$$

The sensitivity with respect to the scaled parameters is given by

$$[\bar{S}]=[D_y][S][D_p]^{-1} \tag{15}$$

Substitution (13) though (15) into (10), leads to

$$\{\bar{p}\}=\{\bar{p}_o\}+[G]\{\Delta\bar{y}\} \tag{16a}$$

$$[G]=[W_{ps}]^{-1}[\bar{S}]^T([\bar{S}][W_{ps}]^{-1}[\bar{S}]^T+[W_{ys}]^{-1})^{-1} \tag{16b}$$

$$[W_{ps}^*]=([I]-[G][\bar{S}])[W_{ps}] \tag{16c}$$

where *Wps* and *Wys* are given by

$$[W_{ps}]=[D_p]^{-1}[W_p][D_p]^{-1},[W_{ys}]=[D_y]^{-1}[W_y][D_y]^{-1} \tag{17}$$

and *Wps* and *Wys* stand for the relative error of the parameters and measurements if *Wp* and *Wy* were the square of the absolute errors of the parameters and measurements.

SCALING BASED ON *Wp*

Different methods to scale the sensitivity matrix are discussed hereafter.

(a) ***Method one****.* A possible choice of *Wp* to scale the sensitivity matrix is to use *Wp** calculated at the previous iteration step. Scaling the sensitivity in this way makes the Bayesian estimator completely equivalent to the Extended Kalman Filter [13, 15]. The advantage of this type of scaling is that the more effective parameters will not be modified much more than the ineffective parameters. This is because in general confidences in effective parameters are higher. Thus, a higher value of *Wp* will be assigned to the more effective parameters so that these parameters will be less modified. The disadvantage of this scaling method is that, it is only valid for the observable system which should be an overdetermined or determined system. For an undetermined system of equations, the convergence criterion will not be satisfied.

(b) ***Method two***. The second possible selection of *Wp* is to use the diagonal terms of *Wp**. This type of scaling uses the Bayesian adaptive nature and the capability to satisfy the convergence criterion in case of an undetermined system. Care should be taken when some ineffective parameter diverges. If for such parameter, *Wp**(*k*)>*Wp**(*k*-1), either fix this parameter, or reduce *Wp**(*k*), forcing it to satisfy the convergence criterion. This method neglected the cross influence among the parameters.

(c) ***Method three***. Test cases showed that constant *Wp* weighting (at all iteration steps, use the same *Wp*) is also very effective for either overdetermined or undetermined systems. Because ineffective parameters will be modified more, the estimation of ineffective parameters is less accurate. Constant weighting *Wp* is able to prevent over-modification of the ineffective parameters. As the weighting *Wp* is constant, it implies that at each iteration step, the variances of the parameter are constant and that the parameters can only be varied between the allowed variances. So, convergence rate is faster than the methods stated in (a) and (b). On the other hand, divergence might occur due to too much modification of the ineffective parameters.

(d) ***Method four****.* In this method, the weighting *Wp* is reduced with each additional iteration according to a certain mathematical relation. For example, we can assume that

$$[W_p(k)]=\alpha[W_p(k-1)]=\alpha^k[W_p(0)] \tag{18}$$

where $0<\alpha<1$.

(e) ***Method five*****.** When setting *Wp* to zero, the weighted least square solution is obtained

$$\{p\}=\{p_o\}+([S]^T[W_y][S])^{-1}[S]^T[W_y]\Delta y \tag{19}$$

The assumption of zero *Wp* is equivalent to an infinite variance: we have no *a priori* knowledge of the parameter values. The direct consequence of this assumption is the large parameter modification that is allowed at each iteration step. Therefore, the possibility of divergence is high. This type of estimator is also referred to as 'improper estimator' [11].

SCALING BASED ON *Wy*

If there exist only one type of responses, the response weighting can easily be determined from the measurement error. As we have stated before, if the error on measured data is high, a low confidence will be assigned resulting in a small parameter modification, and vice versa.

Consider a special case that illustrates the importance of proper scaling. If a measurement is made very accurate it seems reasonable to set $[Wy]^{-1}$ to zero (no measurement noise). By assuming unit weighting *Wp*, (16.b) becomes

$$\{p\} = \{p_o\} + [S]^T([S][S]^T)^{-1}\Delta y \tag{20.a}$$

This is the pseudo solution of an undetermined system. If $[S][S]^T$ contains small diagonal terms, inversion will lead to over-modification of parameter values. This can only be corrected by adding a relatively high $[Wy]^{-1}$ matrix which leads to the odd conclusion that measurements should not be too accurate. In fact, this illustrates that a statistical formulation like the Bayesian, only makes sense when the variance is considerable. Otherwise, deterministic approaches are more appropriate.

Ojalvo et al proposed a so-called ε-decomposition method [16] which differs from the Bayesian estimator in that it introduces an artificial ε-value to solve conditioning problems.

$$\{p\} = \{p_o\} + [S]^T([S][S]^T + \varepsilon)^{-1}\Delta y \tag{20.b}$$

If there is only one type of response, let's say the first ten resonance frequencies of a structure, then different weighting can be selected for each frequency depending on its relative importance. For instance, the lower modes may be more important. Then larger *Wy* can be used for these lower frequencies.

If several types of responses are selected, weighting *Wy* becomes very important, not only related to numerical stability, but also to physical validity. We can rewrite (2) as

$$J(p) = k_f \Delta y_f^T W_f \Delta y_f + k_\varphi \Delta y_\varphi^T W_\varphi \Delta y_\varphi + \ldots$$

$$\ldots + k_m \Delta y_m^T W_m \Delta y_m + k_c \Delta y_c^T W_c \Delta y_c + k_p \Delta p^T W_p \Delta p \tag{21}$$

where subscript f stands for frequency, φ for modes, m for MAC, c for concentrated mass. We can consider three scaling methods: (i) using the norm of response, (ii) using the norm of the sensitivity matrix and (iii) zero weighting.

$$W_f^* = k_f W_f; W_\varphi^* = k_\varphi W_\varphi; W_m^* = k_m W_m; \text{etc.} \tag{22}$$

Weighting by the Norm of Responses.

One way of determining the weighting coefficients is by setting

$$k_f = 1; k_\varphi = \frac{|\Delta y_f^T W_f \Delta y_f|}{|\Delta y_\varphi^T W_\varphi \Delta y_\varphi|} \tag{23}$$

$$k_m = \frac{|\Delta y_f^T W_f \Delta y_f|}{|\Delta y_m^T W_m \Delta y_m|}; k_c = \frac{|\Delta y_f^T W_f \Delta y_f|}{|\Delta y_c^T W_c \Delta y_c|}$$

The coefficients are determined by the Euclidean norm of each type of responses. The physical meaning of the weighting coefficients is that they represent the relative importance among all the selected types of responses.

Weighting by the Norm of the Sensitivity Matrix.

Another way of weighting selection is to use the norm of the sensitivity matrix. The mathematical expression is

$$k_f = 1, k_\varphi = \frac{\left\| \partial y_f / \partial p \right\|}{\left\| \partial y_\varphi / \partial p \right\|}, k_m = \frac{\left\| \partial y_f / \partial p \right\|}{\left\| \partial y_m / \partial p \right\|}, k_c = \frac{\left\| \partial y_f / \partial p \right\|}{\left\| \partial y_c / \partial p \right\|} \tag{24}$$

The sensitivity matrix condition is improved by this type of scaling.

Zero Weighting.

Some responses may be very insensitive to modifications of all the selected parameters. This would be for example the case for element stiffness properties near nodal lines common to all considered mode shapes. In this situation, the Bayesian estimator loses its judgment capability. Then simply put a near zero weighting (variance near infinite) to the response. Such useless responses can be identified by checking the norm of sensitivity

$$S_i = \left\| \partial y_i / \partial p \right\| \tag{25}$$

If for the i-th response, its norm is very small, then a near zero weighting can be assigned to it.

CASE STUDIES

Example 1

A finite element model was updated using different combinations of simultaneous responses

Figure 1 shows the correlation results for three different cases, using the norm of the sensitivity matrix to scale response weighting: (i) using frequencies as responses, (ii) using MAC and frequencies as responses, (iii) using modal displacements and frequencies as responses.

Figure 2 shows the correlation results for two different cases without scaling: (i) using modal displacements and frequencies as responses, and (ii) using MAC and frequencies as responses.

The correlation criterion that is used to evaluate the quality of the updating is the average MAC-value for the mode shapes that are considered, defined as:

$$\text{MeanMAC} = 100 - \tfrac{1}{N}\sum_{i=1}^{N} W_{MAC_i} MAC_i \tag{26}$$

where N is the number of responses.

From figure 1 it can be seen that using MACs or modal displacements as additional responses, the Mean MAC evolution is more stable. When comparing with the results without scaling (figure 2), it is clear that scaling leads to more stable and faster convergence.

Example 2

The first ten resonance frequencies of a cylinder are used as responses to identify four global composite material properties: Young's moduli E_x and E_y, shear modulus G_{xy} and Poissson's ratio ν_{xy}. The covariance matrices were used as *Wp*. Figure 3.a illustrates the parameter modifications using different types of Wp weighting methods. Figure 4.b is the parameter variance evolution using the Kalman filter (Method one). Note that a high conifdence was assigned to ν_{xy} which explains the low parameter modification and variance.

CONCLUSION

Introducing modal displacements, MAC and other responses into the correlation objective function, can improve the quality of model updating results. Stability of the Bayesian parameter estimator when multiple types of responses are used simultaneously, can be improved by different methods to scale the sensitivity matrix. It is important to understand the physical meaning of the scaling methods. A convergence criterion to assess the quality of model updating was proposed.

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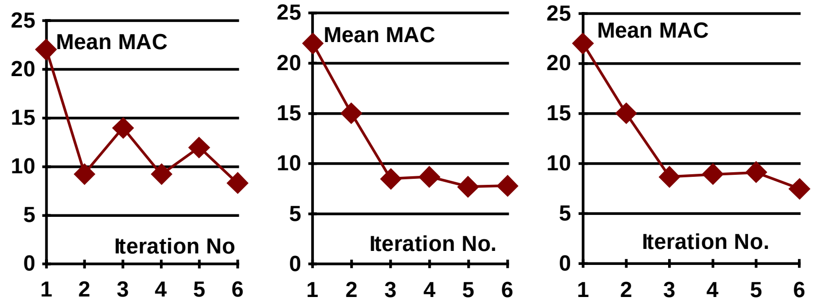
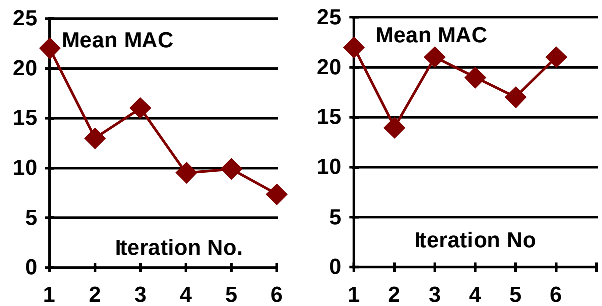


a b c

Figure 1.

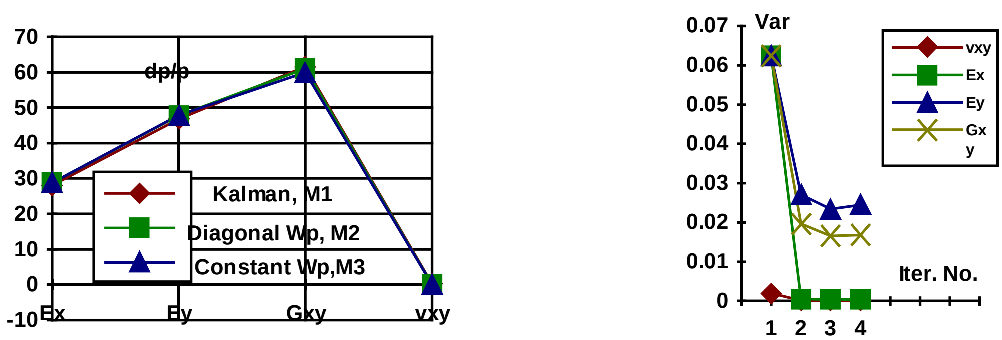
a. Using 12 frequencies as responses with scaling; **b.** Using 12 frequencies and 12 MAC-values as responses with scaling; **c.** Using 12 frequencies and 30 modal displacements as responses with scaling.



a. b.

Figure 2.

a. Using 12 frequencies and 12 MAC as responses without scaling; b. Using 12 frequencies and 30 modal displacements as responses without scaling.



a. Parameter modification (%)

b. Variance evolution

(Kalman filter)

Figure 3.