Updating the Damping Matrix using Frequency Response Data

C. Reix*, C. Tombini**, A Gerard***, J. Strobbe*, E. Dascotte*

*Dynamic Design Solutions
Interleuvenlaan 64
B-3001 Leuven, Belgium

**CEA-CESTA
B.P. 2
33114 Le Barp, France

***University of Bordeaux
251, Cours de Libération
33405 Talence, France

ABSTRACT

This paper presents a method for updating the damping matrix of a linear dynamic system.

For this study, it is assumed that the characteristic mass and stiffness matrices are perfectly known thanks to updating using experimental modal data. Furthermore, it is accepted that damping has only a minor effect on the frequencies and mode shapes of a structure (a hypothesis that is verified for structures with low damping).

It is proposed to adjust the coefficients of the hysteretic damping matrix [D] by superposing the experimental and corresponding analytical Frequency Response Functions (FRF). The frequencies and mode shapes are extracted from the solution of the characteristic equation of movement. An analytical FRF is calculated and then used to compute the sensitivity matrix, showing the influence of the updating parameters on the FRF. To update the damping matrix, a non-linear weighted least squares estimation is used.

NOMENCLATURE

an	subscript for analytical data
exp	subscript for experimental data
e	subscript for estimated data
n	number of degrees of freedom
$[A] \in R^{n*n}$	real matrix with a dimension of n*n
$[B] \in C^{n*n}$	complex matrix with a dimension of n*n
[M]	mass matrix in physical coordinates
[K]	stiffness matrix in physical coordinates
[D]	damping matrix in physical coordinates
[S]	sensitivity matrix
$\{\dot{\ }\}$	first derivative of a vector
$\{\ddot{\ }\}$	second derivative of a vector
λ_j	j^{th} eigenvalue
$\{\varphi\}_j$	j^{th} eigenvector
$\{\psi\}_j$	mass normalized j^{th} eigenvector
H_{ij}	Function Response data , i point of excitation, j point of response.

INTRODUCTION

An important requirement in dynamic analysis is to establish an analytical model capable of reproducing the experimental tests. For this purpose, experimental modal analysis and finite element models that describe the behavior of the structure in terms of frequencies, mode shapes and damping loss factor are often used. It is now possible to adjust the physical element properties that are used in FE models using various approaches [1, 2]. However, to date, only few attempts were made to include the damping matrix. Most FE model updating methods are based on the equation of the conservative model (without damping).

This paper presents an updating method that takes damping into account. The method consists of two stages. Using modern software [3], the first stage consists of adjusting the element material and geometrical properties, and thus implicitly the mass and stiffness matrices. The damping matrix will be updated in the second stage from the superposition of the experimental and analytical Frequency Response Functions (FRFs).

Using FRFs offers several advantages, like for example:

1. The calculation of FRFs is relatively simple, as it is a division of Fourier transforms.
2. FRFs contain all the data such as frequencies, modes shapes and modal damping loss factors. The latter are the most difficult properties to identify.
3. The problem of expansion or reduction of the number of degree of freedom (DOF) does not arise. This problem comes from the fact that the number of experimental DOF is often lower than the number of analytical DOF.
4. It is possible to adjust as many parameters involved in the construction of the damping matrix as there are sampling points on the FRF measurement. This number is often relatively high.

MULTI DEGREE OF FREEDOM SYSTEMS

Damping in the form of a hysteretic (or structural) damping will be considered in the first part. The equation of the movement, for a system with n DOF, subjected to a harmonic excitation, is written as follows :

$$[M]\{\ddot{X}\} + ([K] + i[D])\{X\} = \{f\}\, e^{i\omega t}, \qquad (1)$$
$$[M]\{\ddot{X}\} + [B]\{X\} = \{f\}\, e^{i\omega t},$$
$$i = \sqrt{-1}\ .$$

Considering the homogenous equation without the second member, solutions of (1) take the following form:

$$\{X\} = \{\phi\}\, e^{i\mu t} \qquad (2)$$

Substituting (2) into (1), the characteristic equation is obtained:

$$(-\mu_r^2[M] + [B])\{\phi\} = \{0\} \Leftrightarrow (-\lambda_r[M] + [B])\{\phi\} \qquad (3)$$

The solutions appear in the form of two complex matrices: one of dimension (n x 1) containing the eigenvalues and the other of dimension (n x n), where the columns contain the mode shapes $\{\varphi_r\}$ associated with the eigenvalues μ_r :

$$\mu_r^2 = \omega_r^2\,(1 + i.\eta_r) \qquad (4)$$

where ω_r represent the natural frequency, and η_r the damping loss factor. The mass and the [B] matrices, written in the base of the mode shapes, are diagonal and

$$[\varphi]^T[M][\varphi] = \begin{bmatrix} \ddots & & 0 \\ & m & \\ 0 & & \ddots \end{bmatrix}, [\varphi]^T[K + i\,D][\varphi] = \begin{bmatrix} \ddots & & 0 \\ & k & \\ 0 & & \ddots \end{bmatrix}, \qquad (5)$$

and

$$\mu_r^2 = \frac{k_r}{m_r}\ .$$

The mass-normalized mode shape for mode r is:

$$\{\psi\}_r = \frac{1}{\sqrt{m_r}}\{\varphi\}_r. \qquad (6)$$

Considering the particular case where the excitation and the response to equation (1) are respectively a harmonic response and excitation, this yields:

$$\{\phi\} = ([K] + i[D] - \omega^2[M])^{-1}\{f\} = [h(\omega)]\{f\}\ . \qquad (7)$$

from which the matrix of the following FRFs is derived:

$$[h(\omega)] = [\psi]\begin{bmatrix} \ddots & & 0 \\ & (\mu_r^2 - \omega^2) & \\ 0 & & \ddots \end{bmatrix}^{-1}[\psi]^T\ . \qquad (8)$$

From this matrix, a receptance is extracted such that, if the excitation occurs at a point i and the response at a point j (or vice versa), the following equation is obtained:

$$h_{ij}(\omega) = \sum_{r=1}^{n}\frac{\psi_{ri}\psi_{rj}}{\omega_r^2 - \omega^2 + i\eta_r\omega_r^2} = \sum_{r=1}^{n}\frac{\psi_{ri}\psi_{rj}}{\lambda_r - \omega^2} \qquad (9)$$

The accelerance (or inertance) becomes:

$$H_{ij}(\omega) = -\omega^2 h_{ij}(\omega) \qquad (10)$$

CONSTRUCTION OF THE FE MODEL

The structure is discretized into N finite elements and the mass and stiffness matrices are computed such that:

$$[M] = \sum_{i=1}^{N} M_{ie}$$
$$[K] = \sum_{i=1}^{N} K_{ie} \qquad (11)$$

Let

$$[D] = \sum_{i=1}^{N} \alpha_i K_{ie} + \beta_i M_{ie} \qquad (12)$$

For an homogenous, isotropic and linear structure, this gives:

$$[D] = \alpha K + \beta M \qquad (13)$$

In this particular case, the mode shapes are found to be identical to those of the conservative system, and the eigenvalues are written in the following complex form:

$$\mu_r^2 = \overline{\omega}_r^{\,2}\,(1 + i.\eta_r) \qquad (14)$$

where $\overline{\omega}_r$ are the frequencies of the non-damped system.

PROCEDURE FOR FRF-BASED UPDATING

The method consists of superposing an experimental FRF on an analytical FRF. The analytical FRF is calculated from equation (10), by attributing arbitrary values to α_i and β_i. Using successive iterations, and by calculating the sensitivity of the FRF with respect to the α_i and β_i coefficients, the values that minimize the difference between the experimental FRF and the analytical FRF are obtained.

Let R be the vector of the parameters to be adjusted:

$$R = \begin{Bmatrix} r_1 \\ \vdots \\ r_p \end{Bmatrix} = \begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} \qquad (15)$$

Sensitivity matrix [S] translates the influence of the parameters r on the FRF:

$$[S] = \left[\frac{\partial H_{ij}}{\partial r}\right] = \left[\frac{\partial H_{ij}}{\partial \lambda}\right]\left[\frac{\partial \lambda}{\partial r}\right] + \left[\frac{\partial H_{ij}}{\partial \psi}\right]\left[\frac{\partial \psi}{\partial r}\right]$$

$$[S] = \left[\frac{\partial H_{ij}}{\partial \lambda}\right]\left\{\left[\frac{\partial \lambda}{\partial m}\right]\left[\frac{\partial m}{\partial r}\right] + \left[\frac{\partial \lambda}{\partial b}\right]\left[\frac{\partial b}{\partial r}\right]\right\} + \left[\frac{\partial H_{ij}}{\partial \psi}\right]\left\{\left[\frac{\partial \psi}{\partial m}\right]\left[\frac{\partial m}{\partial r}\right] + \left[\frac{\partial \psi}{\partial b}\right]\left[\frac{\partial b}{\partial r}\right]\right\}$$

$$[S] = \left[\frac{\partial H_{ij}}{\partial \lambda}\right]\left[\frac{\partial \lambda}{\partial b}\right]\left[\frac{\partial b}{\partial r}\right] + \left[\frac{\partial H_{ij}}{\partial \psi}\right]\left[\frac{\partial \psi}{\partial b}\right]\left[\frac{\partial b}{\partial r}\right] \qquad (16)$$

with

$$\left[\frac{\partial m}{\partial r}\right] = [0]$$

The measurement points are noted ω_i, for i = 1 to q. The derivatives of the FRF with respect to the eigenvalues and mode shapes take the following forms:

$$\left[\frac{\partial H_{ij}}{\partial \lambda}\right]=\begin{bmatrix}\frac{\partial H_{ij}(\omega_1)}{\partial \lambda_1} & \cdots & \frac{\partial H_{ij}(\omega_1)}{\partial \lambda_n}\\ \vdots & \ddots & \vdots\\ \frac{\partial H_{ij}(\omega_q)}{\partial \lambda_1} & \cdots & \frac{\partial H_{ij}(\omega_q)}{\partial \lambda_n}\end{bmatrix} \tag{17}$$

$\left[\frac{\partial H_{ij}}{\partial \lambda}\right]$ is a rectangular matrix (q x n), with

$$\left[\frac{\partial H_{ij}(\omega)}{\partial \lambda_k}\right]=\frac{\omega^2\psi_{ik}\psi_{jk}}{(\lambda_k-\omega^2)^2} \tag{18}$$

Furthermore:

$$\left[\frac{\partial H_{ij}}{\partial \psi}\right]=\begin{bmatrix}\frac{\partial H_{ij}(\omega_1)}{\partial \psi_{11}} & \cdots & \frac{\partial H_{ij}(\omega_1)}{\partial \psi_{nn}}\\ \vdots & \ddots & \vdots\\ \frac{\partial H_{ij}(\omega_q)}{\partial \psi_{11}} & \cdots & \frac{\partial H_{ij}(\omega_q)}{\partial \psi_{nn}}\end{bmatrix} \tag{19}$$

$\left[\frac{\partial H_{ij}}{\partial \psi}\right]$ is a rectangular matrix (q x n^2). Let

$$r = E\left(\frac{k}{n}\right)+1 \tag{20}$$

E(x) designating the entire part of x, and s = k + 1 - n (r - 1),

for k = 0 to n^2 - 1

$$\frac{\partial H_{ij}(\omega)}{\partial \psi_{rs}}=\begin{Bmatrix}\text{if } i=j=r,\ -\omega^2\,\frac{2\ \psi_{rs}}{(\lambda_s-\omega^2)}\\ \text{if } i=r \text{ and } j\neq r,\ -\omega^2\,\frac{\psi_{js}}{(\lambda_s-\omega^2)}\\ \text{if } \neq r \text{ and } j=r,\ -\omega^2\frac{\psi_{is}}{(\lambda_s-\omega^2)}\\ \text{if } i\neq r \text{ and } j\neq r,\ 0\end{Bmatrix} \tag{21}$$

Furthermore, the matrices of the derivatives of the eigenvalues and of the mode shapes with respect to the b coefficients of the [B] matrix give [4]:

$$\left[\frac{\partial \lambda}{\partial b}\right]=\begin{bmatrix}\frac{\partial \lambda_1}{\partial b_{11}} & \cdots & \frac{\partial \lambda_1}{\partial b_{nn}}\\ \vdots & \ddots & \vdots\\ \frac{\partial \lambda_n}{\partial b_{11}} & \cdots & \frac{\partial \lambda_n}{\partial \alpha_{nn}}\end{bmatrix} \tag{22}$$

with

$$\frac{\partial \lambda_i}{\partial b_{rs}}=\psi_{rs}\psi_{si} \tag{23}$$

$$\left[\frac{\partial \psi}{\partial b}\right]=\begin{bmatrix}\frac{\partial \psi_{11}}{\partial b_{11}} & \cdots & \frac{\partial \psi_{11}}{\partial b_{nn}}\\ \vdots & \ddots & \vdots\\ \frac{\partial \psi_{n,n}}{\partial b_{11}} & \cdots & \frac{\partial \psi_{n,n}}{\partial \alpha_{nn}}\end{bmatrix} \tag{24}$$

$$\frac{\partial \psi_{ij}}{\partial \psi_{kl}}=\sum_{m=1}^{n}\frac{(1-\delta_{mj})\lambda_j\psi_{im}\psi_{km}\psi_{lj}}{\lambda_j-\lambda_m} \tag{25}$$

where

$$\delta_{mj}=1 \quad \text{if } m=j$$

$$\delta_{mj}=0 \quad \text{if } m\neq j$$

$\left[\frac{\partial \psi}{\partial b}\right]$ is a square martix (n^2 x n^2).

The final stage of the calculation of the sensitivity matrix consists of calculating the derivative of the matrix [B] with respect to vector R:

$$\left[\frac{\partial b}{\partial r}\right]=\begin{bmatrix}\frac{\partial b_{11}}{\partial r_1} & \cdots & \frac{\partial b_{11}}{\partial r_p}\\ \vdots & \ddots & \vdots\\ \frac{\partial b_{nn}}{\partial r_1} & \cdots & \frac{\partial b_{nn}}{\partial r_p}\end{bmatrix} \tag{26}$$

$\frac{\partial b}{\partial r}$ is a rectangular matrix (n^2 x p)

$$\frac{\partial b_{lj}}{\partial r}=\left[i[K_{ke}],i[M_{ke}]\right] \qquad i=\sqrt{-1} \tag{27}$$

A non-linear weighted least squares resolution [5] is used to minimize function F, with:

$$F=\sum_{i=1}^{q}w_{ii}\left\|\left(H_{an}(\omega_i)-H_{exp}(\omega_i)\right)\right\|^2 \tag{28}$$

with H (ω_i) = H (ω_i, r_1, r_2, $\cdots$, r_p).

w_i is a weight which is equal to the inverse of square of the uncertainty where:

$$w_{ii}=\frac{1}{\sigma_i^2} \tag{29}$$

which results in solving:

$$\sum_{i=1}^{q}w_{ii}\left(H_{an}(\omega_i)-H_{exp}(\omega_i)\right)\frac{\partial H(\omega_i)}{\partial r_k}=0 \tag{30}$$

with k=1 to p.

To obtain a linear formulation, H_{an} is written in the form of a Taylor series development of the first order:

$$H_{an}=H_{ane}+(r_1-r_{1e})\left(\frac{\partial H_{an}}{\partial r_1}\right)_{r=r1}+\cdots+(r_p-r_{pe})\left(\frac{\partial H_{an}}{\partial r_p}\right)_{r=rp}$$

(31)

If

$$R_l = (r_l - r_{le}), \tag{32}$$

and let [S] be the sensitivity matrix:

$$S_{ij} = \left[\frac{\partial H_{an}(\omega_i)}{r_j}\right] \quad \text{for } i = 1 \text{to } q, \text{ and } j = 1 \text{ to } p. \tag{33}$$

then the introduction of (31), (32) and (33) in (30) gives:

$$\sum_{i=1}^{q} w_{ii}\left(H_{exp}(\omega_i) - H_{ane}(\omega_i) - R_1 S_{i1} - \ldots - R_p S_{ip}\right) S_{ik} = 0 \tag{34}$$

for k = 1 to p.

Expression (34) is equivalent to solving the following linear system:

$$[E][R] = [V] \tag{35}$$

with

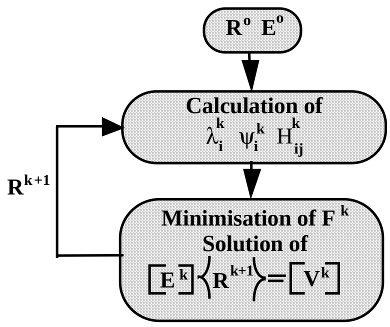
$$\begin{aligned} [E] &\in C^{pxp} \\ [R] &\in R^{px1} \\ [V] &\in C^{px1} \end{aligned} \tag{36}$$

$$E_{jk} = \sum_{i=1}^{q} w_{ii} S_{ij} S_{ik} \tag{37}$$

$$V_j = \sum_{i=1}^{q} w_{ii} S_{ij}\left(H_{exp}(\omega_i) - H_{ane}(\omega_i)\right) \tag{38}$$

Note that it is easier to update the parameters on the modulus of the FRF (and/or the phase), as in this case the matrix E and the vector V are real. The solution R will therefore necessarily be real.

The different steps of the algorithm are summarized in the following flow-chart:



APPLICATION EXAMPLE

The method is applied to a bar made of plexiglass material for which the modes of traction and compression are of interest. The structure is modeled using 12 bar elements with one degree of freedom per node (figure 1). After assembly, the mass and stiffness matrices are tri-diagonal in blocks.

Experimentally, the structure is excited at one of its ends. The response signal is recorded at the other end (figure 2). By modal analysis, the first three frequencies and the first three mode shapes in traction-compression are extracted.

The first step consists of updating the Young's modulus of the material for all elements (its density being perfectly known) from the comparison of experimental and analytical resonance frequencies and modes shapes.

The following table shows the experimental and analytical frequencies after updating the Young's modulus E:

Mode Number	Analytical Freq. (Hz)	Experim. Freq (Hz)
1	887	887
2	1784	1791
3	2700	2696

Table 1: Resonance frequencies of the plexiglass bar.

The second step consists of using the method previously developed. Updating will be performed on the α et β coefficients such that $\alpha = \alpha_i$ and $\beta = \beta_i$, irrespective of i. Arbitrary starting values are given to these two coefficients. After five iterations, the analytical FRF converges towards the experimental FRF (figures 3 and 4).

CONCLUSION

A method to iteratively identify damping coefficients from experimentally obtained FRFs was presented. From initial applications, it was observed that convergence of the method is better when the starting values for α et β coefficients are not too far from the solution. Convergence then requires only a few number of iterations.

To verify that the FRF that is calculated from by the analytical model is sufficiently close to the experimental one, the Signature Assurance Criterion (SAC) [6] between these two FRFs is used:

$$SAC(u, v) = 100 \frac{\left|u^T v\right|^2}{(u^T u v^T v)} \tag{37}$$

Two perfectly correlated FRFs correspond to a SAC of 100. Obtaining a SAC of 100 is a necessary but not sufficient criterion (because, for example, of an insufficient sampling).

The matrices that are used in the construction of sensitivity matrix [S] can reach large dimensions (up to a maximum of $n^2 * n^2$ where n is the number of DOF of the FE model). This consideration must be taken into account when programming the method.

The method is very promising and was implemented as an extension to the FemTools software program [3]. Further ongoing work consists of defining bounds of applicability by using it with different types of real life structures like assembled structures in order to identify damping at the joints.

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FIGURES

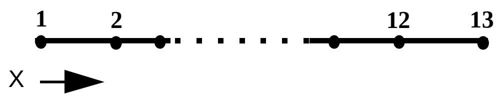

Figure 1: Model of the beam.

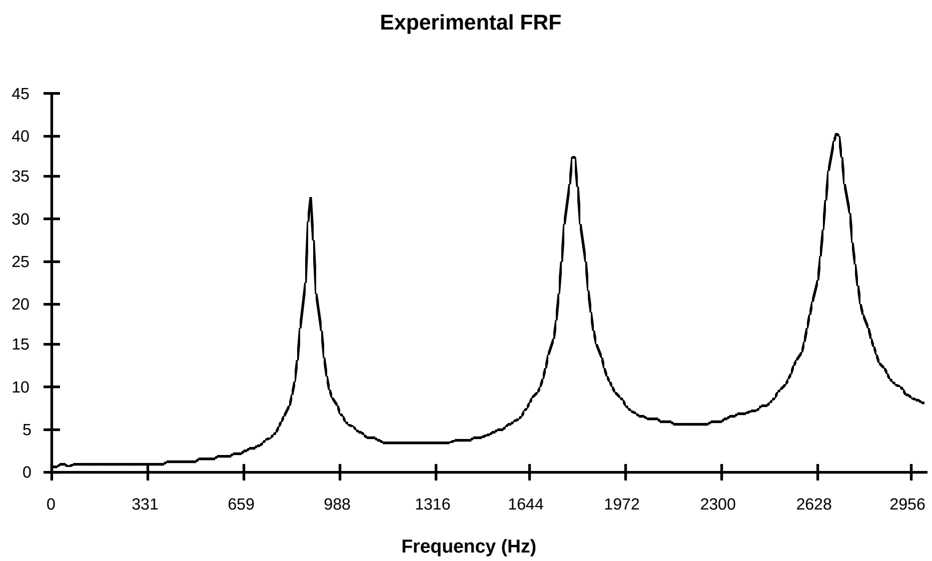

Figure 2 : Experimental FRF data of the beam for traction-compression.

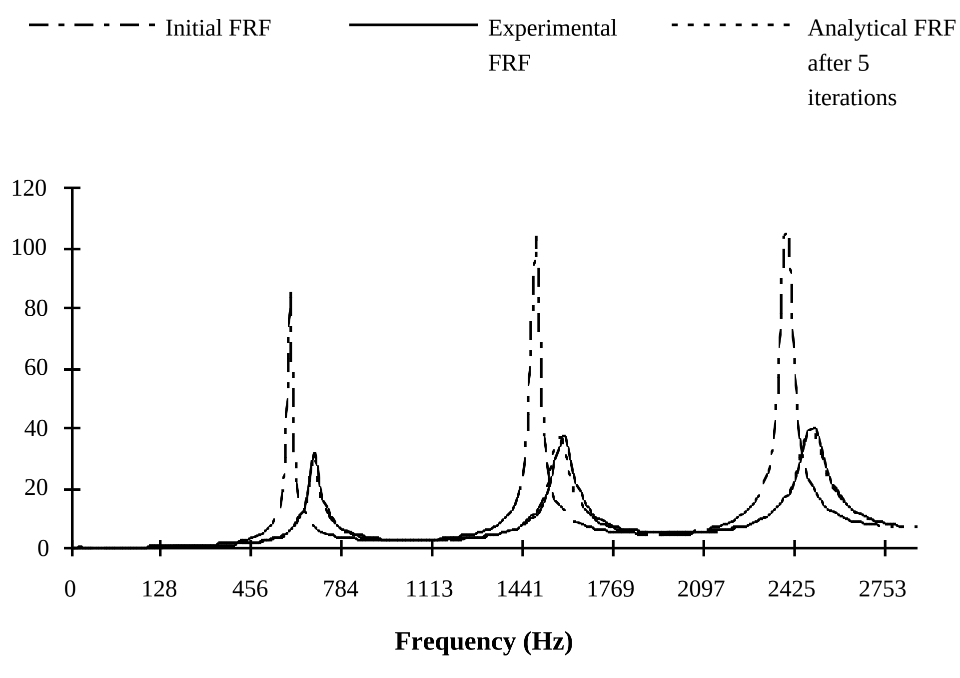

Figure 3: Superposition of the experimental FRF and the updated FRF after five iterations.

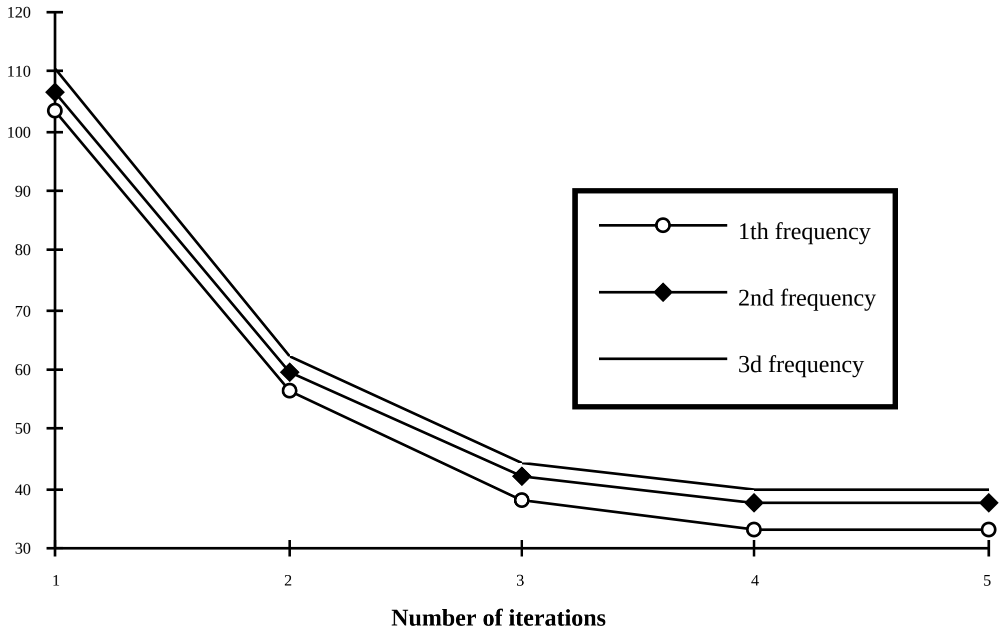

Figure 4: Convergence of the magnitudes of the analytical FRF for the first 3 frequencies.